FIBRATIONS OF STRICT n-GROUPOIDS

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Definition 0.1. *A k-morphism $f : x \to y$ in a strict n-category is a **k-equivalence** if there exists $g : y \to x$ and $(k+1)$-equivalences $f \circ g \to \mathrm{id}_y$, $g \circ f \to \mathrm{id}_x$.*

Definition 0.2. *A **strict n-groupoid** is a strict n-category all of whose k-morphisms are k-equivalences.*

Notation 0.3. *$\theta^{(k)}$ is the free walking k-cell, denoted O^k in* [1].
I^k is the free walking coherent k-equivalence, denoted P^{k-1} in [1].

Definition 0.4. *A map $f : X \to Y$ between strict n-groupoids is a **folk fibration** if it is a fibration in the folk model structure on strict n-categories of* [1]. *This means every lifting problem of the form*

$$\begin{array}{ccc} \theta^{(k)} & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ I^{k+1} & \longrightarrow & Y \end{array}$$

has a solution.

Definition 0.5. *A map $f : X \to Y$ between strict n-groupoids is a **fibration** if every lifting problem of the form*

$$\begin{array}{ccc} \theta^{(k)} & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ \theta^{(k+1)} & \longrightarrow & Y \end{array}$$

has a solution.

Lemma 0.6. *Let X be an n-category and $u : x \to x'$ a k-equivalence in X. Then there exists $v : x \to x'$, which admits an extension*

$$\begin{array}{ccc} \theta^{(k)} & \xrightarrow{v} & X \\ \downarrow & \nearrow & \\ I^k & & \end{array},$$

and a $(k+1)$-equivalence $u \to v$.

Proof. The existence of v with the extension property is guaranteed by Lemma 18 in [1]. By examining the proof, one sees that it actually also provides the equivalence $u \to v$.

□

Proposition 0.7. *Let X, Y be n-groupoids and $f : X \to Y$ a folk fibration. Then f is a fibration.*

Proof. Consider the lifting problem

$$\begin{array}{ccc} \theta^{(k)} & \longrightarrow & X \\ \big\downarrow & \overset{?}{\nearrow} & \big\downarrow f \\ \theta^{(k+1)} & \xrightarrow[u]{} & Y. \end{array}$$

Let $v \simeq u$ such that the extension

$$\begin{array}{ccc} \theta^{(k+1)} & \xrightarrow{v} & Y \\ \big\downarrow & \nearrow & \\ I^{k+1} & & \end{array}$$

exists. Because f is a folk fibration, we can solve the lifting problem

$$\begin{array}{ccc} \theta^{(k)} & \longrightarrow & X \\ \big\downarrow & \overset{\exists}{\nearrow} & \big\downarrow f \\ I^{k+1} & \longrightarrow & Y. \end{array}$$

Now we restrict this lift to $\theta^{(k+1)}$ to obtain

$$\begin{array}{ccc} \theta^{(k)} & \longrightarrow & X \\ \big\downarrow & \overset{\tilde{v}}{\nearrow} & \big\downarrow f \\ \theta^{(k+1)} & \xrightarrow[v]{} & Y. \end{array}$$

Let $\varphi : v \to u$ be the above mentioned equivalence. By induction, we can solve the following lifting problem.

$$\begin{array}{ccc} \theta^{(k+1)} & \xrightarrow{\tilde{v}} & X \\ \big\downarrow & \overset{\exists}{\nearrow} & \big\downarrow f \\ \theta^{(k+2)} & \xrightarrow[\varphi]{} & Y. \end{array}$$

The lift now includes a lift of $u = t(\varphi)$.

□

References

[1] Yves Lafont, François Métayer, and Krzysztof Worytkiewicz. A folk model structure on omega-cat. *Advances in Mathematics*, 224(3):1183–1231, 2010.